

# A Rotating and Radiating Metric

Sanjay M. Wagh<sup>†</sup> and Pradeep S. Muktibodh<sup>†‡</sup>

<sup>†</sup>Central India Research Institute, Post Box 606, Laxminagar, Nagpur 440 022, India

E-mail : ciri@bom2.vsnl.net.in

<sup>‡</sup>Department of Mathematics, Hislop College, Temple Road, Civil Lines, Nagpur 440 001, India

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## Abstract

A non-static solution of Einstein's field equations of General Relativity representing the gravitational field of an axisymmetric radiation flow is obtained using the Eddington or the Kerr-Schild form for the metric. A solution obtained here manifestly corresponds to the Kerr metric with its mass-parameter,  $m$ , being an arbitrary function of the advanced (retarded) null-time coordinate. Then, when  $m$  is constant, the solution reduces to the standard Kerr metric expressed in terms of the used null coordinate. And, when the angular momentum parameter,  $a$ , constant here, is set to zero, the solution reduces to the Vaidya metric expressed using the corresponding null coordinate.

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*Running head:* Rotating and Radiating Metric —

# 1 Introduction

A non-static generalization of the Schwarzschild metric describing the gravitational field of a radiating star or that of flowing radiation is the well-known Vaidya metric [1]. As is equally well-known [2], the Kerr metric [3] describes the exterior field outside of a rotating, axisymmetric black hole. In fact, it is [3] the only known stationary, asymptotically flat, vacuum metric possessing a central gravitating black hole and inherent spacetime rotation. Then, the next logical step is that of obtaining a suitable non-static generalization of the Kerr metric which would correspond to the exterior gravitational field of a rotating and radiating body. This problem attracted the attention of several workers in the past [4]. However, to the best of our knowledge, none of the reported solutions can be considered entirely satisfactory. For example, it is difficult to study the axisymmetric collapse of radiation shells using these metrics.

Therefore, it is the purpose of the present paper to investigate this problem anew. Apart from the obvious purpose of extending the class of known solutions to the field equations of General Relativity, aspects related to other important problems such as those related to the axisymmetric collapse of radiation shells and the Cosmic Censorship [5] are the real motivation behind the present work. However, the relevance of the radiating and rotating metric presented here for such problems will be the subject of an independent series of publications [6].

In the present paper, we base our derivation of the radiating and rotating metric on the Eddington or the Kerr-Schild form [2]. It will be clear from our procedure of derivation that the solution obtained here manifestly corresponds to the Kerr metric with its mass-parameter,  $m$ , being an arbitrary function of the space-time coordinates. Thus, when  $m$  is constant, the solution is the Kerr metric and, when the angular momentum parameter,  $a$ , constant here, is set to zero the solution reduces to the Vaidya metric [1]. [We use  $G = c = 1$  units unless explicitly mentioned otherwise.]

## 2 Metric, Its Properties and Field Equations

We begin our derivation with the following metric of the Eddington or the Kerr-Schild form

$$g_{\alpha\beta} = \eta_{\alpha\beta} - 2\ell_{\alpha}\ell_{\beta} \tag{1}$$

where  $\eta_{\alpha\beta}$  is the flat Lorentzian metric in Cartesian coordinates and  $\ell_\alpha$  is a null-vector with respect to  $\eta_{\alpha\beta}$ . Defining  $\ell^\alpha = \eta^{\alpha\beta}\ell_\beta$ , the inverse metric  $g^{\alpha\beta}$  can be easily shown to be

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2\ell^\alpha\ell^\beta \quad (2)$$

It then follows that the indices of  $\ell_\alpha$  can be raised or lowered using either  $\eta_{\alpha\beta}$  or  $g_{\alpha\beta}$ . Further,

$$\ell^\alpha{}_{,\beta}\ell_\alpha = 0 = \ell_{\alpha,\beta}\ell^\alpha \quad (3)$$

where a comma denotes an ordinary derivative. Then, it can be shown that

$$\left\{ \begin{matrix} \alpha \\ \beta \quad \tau \end{matrix} \right\} \ell^\tau = -\ell^\alpha{}_{,\tau}\ell^\tau\ell_\beta - \ell^\alpha\ell_{\beta,\tau}\ell^\tau \quad (4)$$

which implies that

$$L = -\ell^\alpha{}_{;\alpha} \equiv -\ell^\alpha{}_{,\alpha} \quad (5)$$

$$V^\alpha\ell_\alpha = 0 \quad V^\alpha = \ell^\alpha{}_{;\beta}\ell^\beta \equiv \ell^\alpha{}_{,\beta}\ell^\beta \quad (6)$$

where  $\left\{ \begin{matrix} \alpha \\ \beta \quad \tau \end{matrix} \right\}$  is the usual Christoffel symbol of the second kind and the semicolon denotes the metric preserving covariant derivative.

Moreover, it can be shown [2] that  $\sqrt{-g} = 1$  quite generally, where  $g$  denotes the determinant of the metric. Therefore,

$$\left\{ \begin{matrix} \alpha \\ \beta \quad \alpha \end{matrix} \right\} = [\log \sqrt{-g}]_{,\beta} = 0$$

and the Ricci tensor is

$$\begin{aligned} R_{\beta\delta} &= \left\{ \begin{matrix} \alpha \\ \beta \quad \alpha \end{matrix} \right\}_{,\delta} - \left\{ \begin{matrix} \alpha \\ \beta \quad \delta \end{matrix} \right\}_{,\alpha} + \left\{ \begin{matrix} \alpha \\ \tau \quad \delta \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \beta \quad \alpha \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \tau \quad \alpha \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \beta \quad \delta \end{matrix} \right\} \\ &\equiv -\left\{ \begin{matrix} \alpha \\ \beta \quad \delta \end{matrix} \right\}_{,\alpha} + \left\{ \begin{matrix} \alpha \\ \tau \quad \delta \end{matrix} \right\} \left\{ \begin{matrix} \tau \\ \beta \quad \alpha \end{matrix} \right\} \end{aligned} \quad (7)$$

Now, we assume that

$$V^\alpha V_\alpha = 0 \quad (8)$$

Then, it can be shown [2] that  $V^\alpha$  and  $\ell^\alpha$  must be proportional to each other. And, we write

$$V^\alpha \equiv \ell^\alpha{}_{,\beta} \ell^\beta = -A(x^\tau) \ell^\alpha \quad (9)$$

where  $A(x^\tau)$  is a scalar field. We note that the indices of  $V^\alpha$  can be raised or lowered using either  $\eta_{\alpha\beta}$  or  $g_{\alpha\beta}$ .

Further, we use the field equations of general relativity in the following form

$$R_{\beta\delta} = \kappa T_{\beta\delta} - \frac{\kappa T}{2} g_{\beta\delta} \quad \kappa = \frac{-8\pi G}{c^4} \quad (10)$$

where  $G$  is Newton's gravitational constant and  $T_{\beta\delta}$  is the energy-momentum tensor whose trace is denoted by  $T$ .

Now, using properties (3), (5), (6), (8) and (9) and rearranging terms, we obtain

$$\begin{aligned} R_{\beta\delta} = & - (\ell_\beta \ell_\delta)_{,\alpha\mu} \eta^{\alpha\mu} - [(L+A)\ell_\beta]_{,\delta} - [(L+A)\ell_\delta]_{,\beta} \\ & + 2[2(A\ell^\alpha)_{,\alpha} - A^2 + \ell^\mu{}_{,\nu} \ell^\nu{}_{,\mu} - \eta^{\alpha\mu} \ell^\nu{}_{,\mu} \ell_{\nu,\alpha}] \ell_\beta \ell_\delta \end{aligned} \quad (11)$$

Multiplying and contracting eq. (11) with  $\ell^\beta$ , expanding the terms and using properties (3), (5), (6), (8), (9) and eq. (10), we obtain

$$\begin{aligned} \ell^\beta R_{\beta\delta} &= \kappa \ell^\beta T_{\beta\delta} - \frac{\kappa T}{2} \ell_\delta \\ &= [AL + A^2 - (L+A)_{,\alpha} \ell^\alpha - \ell^\beta \ell_{\beta,\alpha\mu} \eta^{\alpha\mu}] \ell_\delta \end{aligned} \quad (12)$$

But, using the above mentioned properties, it can be easily seen that

$$\begin{aligned} [2(A\ell^\alpha)_{,\alpha} - A^2 + \ell^\mu{}_{,\nu} \ell^\nu{}_{,\mu} - \eta^{\alpha\mu} \ell^\nu{}_{,\mu} \ell_{\nu,\alpha}] \ell_\delta \\ = -[AL + A^2 - (L+A)_{,\alpha} \ell^\alpha - \ell^\nu \ell_{\nu,\alpha\mu} \eta^{\alpha\mu}] \ell_\delta \\ = -\ell^\alpha R_{\alpha\delta} = -\kappa \ell^\alpha T_{\alpha\delta} + \frac{\kappa T}{2} \ell_\delta \end{aligned} \quad (13)$$

Then, using eqs. (11) and (13) in eq. (10), we obtain

$$\begin{aligned} -(\ell_\beta \ell_\delta)_{,\alpha\mu} \eta^{\alpha\mu} - [(L+A)\ell_\beta]_{,\delta} - [(L+A)\ell_\delta]_{,\beta} \\ = \kappa(T_{\beta\delta} - \frac{T}{2} g_{\beta\delta}) + 2\kappa(\ell^\alpha T_{\alpha\delta}) \ell_\beta \end{aligned} \quad (14)$$

With a choice of the energy-momentum tensor appropriate to the problem at hand, we now turn to solving the field equations (14).

### 3 Equations for the Radiating Metric

Now, let the *directed flow of radiation* mean a distribution of electromagnetic energy which any chosen local observer finds flowing in one and only one direction at that chosen location. Then, the energy-momentum tensor for such a radiation field, as is well-known, is given by

$$T_{\beta\delta} = \zeta \ell_\beta \ell_\delta \quad T \equiv T^\alpha_\alpha = 0 \quad \zeta = \frac{\sigma}{\ell_o^2} \quad (15)$$

where  $\sigma$  is the density of radiation and the lines of flow of radiation are the null geodesics of the metric  $g_{\alpha\beta}$ . It may be noted that  $\ell_o$  is a dimensionless quantity and, hence,  $\zeta$  and  $\sigma$  have the dimensions of energy density, both. However, as will be seen later, it is  $\sigma$  that is directly related to the flux of radiation.

Then, for the traceless energy-momentum tensor of eq. (15), we note that  $\ell^\alpha T_{\alpha\delta} = 0$ . Therefore, the field equations, eq. (14), become

$$-\square^2(\ell_\beta \ell_\delta) - [(L + A)\ell_\beta]_{,\delta} - [(L + A)\ell_\delta]_{,\beta} = \frac{\kappa\sigma}{\ell_o^2} \ell_\beta \ell_\delta \quad (16)$$

where  $\square^2 = (\partial^2/\partial x^{o2}) - \nabla^2$  is the standard D'Alembertian operator.

Now, introduce a three-vector  $\lambda_i$  as (latin indices to take values 1, 2, 3)

$$\ell_\alpha = \ell_o(1, \lambda_i), \quad \ell^\alpha = \ell_o(1, -\lambda^i) \quad (17)$$

such that  $\lambda_i$  is a flat-space unit vector,  $\lambda^i \lambda_i = 1$  since  $\ell^\alpha$  is a flat-space null vector. Expressed in terms of  $\lambda_i$ , the field equations are

$$\begin{aligned} (\beta = \delta = 0) \\ -\square^2(\ell_o^2) - 2[(L + A)\ell_o]_{,o} = \kappa\sigma \end{aligned} \quad (18A)$$

$$\begin{aligned} (\beta = 0, \delta = i) \\ -\square^2(\ell_o^2 \lambda_i) - [(L + A)\ell_o]_{,i} - [(L + A)\ell_o \lambda_i]_{,o} = \kappa\sigma \lambda_i \end{aligned} \quad (18B)$$

$$\begin{aligned} (\beta = i, \delta = j) \\ -\square^2(\ell_o^2 \lambda_i \lambda_j) - [(L + A)\ell_o \lambda_i]_{,j} - [(L + A)\ell_o \lambda_j]_{,i} = \kappa\sigma \lambda_i \lambda_j \end{aligned} \quad (18C)$$

Now, using the explicit form of the D'Alembertian and

$$\lambda_{i,o} = 0 \quad (19)$$

the eq. (18C) can be manipulated with the help of eqs. (18A) and (18B) to yield

$$\lambda_{i,k} \lambda_{j,k} = \left( \frac{L+A}{2\ell_o} \right) [\lambda_{i,j} + \lambda_{j,i}] \quad (20)$$

Here and hereafter, we sum over any repeated latin index regardless of position, for example, index  $k$  is to be summed over in eq. (20).

Now, using eq. (5) and eq. (6) with  $\alpha = 0$ , we get

$$L + A = \ell_o \lambda_{i,i} + 2 \ell_{o,i} \lambda_i - 2 \ell_{o,o} \quad (21)$$

The form of the last two terms of eq. (21) then suggests a substitution

$$\ell_o^2 = m(x^\alpha) g(x^k) \quad (22)$$

which gives

$$\frac{L+A}{\ell_o} = \lambda_{i,i} + \frac{g_{,i} \lambda_i}{g} + \frac{1}{m} [m_{,i} \lambda_i - m_{,o}] \quad (21A)$$

Demanding  $(L+A)/\ell_o$  to be independent of  $m$ , we then require

$$m_{,o} = m_{,i} \lambda_i \quad (23)$$

so that

$$\frac{L+A}{\ell_o} = \lambda_{i,i} + \frac{g_{,i} \lambda_i}{g} \equiv 2p(x^k), \quad \text{say} \quad (21B)$$

Then, eq. (20) is the same as the one which arises in the stationary case,  $m = \text{constant}$  and can be solved in the manner as in [2]. Therefore, without going into the details, which can be found in [2], we note that the field eq. (20) is equivalent to

$$\nabla^2 \gamma = 0, \quad (\nabla \omega)^2 = 1 \quad (24)$$

where

$$\gamma = \alpha + i\beta, \quad \alpha = p(1 - \cos \vartheta), \quad \beta = p \sin \vartheta, \quad \omega = \frac{1}{\gamma} \quad (25)$$

( Here  $\vartheta$  is the angle of rotation which brings  $\lambda_i$  along the  $X$ -axis. ) Depending upon consistent boundary conditions the above Laplace and the Eikonal equations, eq. (24), can be solved and the unit vector  $\vec{\lambda}$  can be obtained as

$$\vec{\lambda} = \frac{\nabla\omega + \nabla\omega^* - \imath(\nabla\omega \times \nabla\omega^*)}{1 + \nabla\omega \bullet \nabla\omega^*} \quad (26)$$

which satisfies eq.(20) or equivalently,

$$\lambda_{i,k} = \alpha (\delta_{ik} - \lambda_i \lambda_k) + \beta \epsilon_{ikl} \lambda_l \quad (27)$$

Moreover, we also note at this point that when  $m = \text{constant}$ ,

$$g(x^k) = \alpha \quad (28)$$

is the solution to field equations, eqs. (18A) - (18C), unique upto a multiplicative constant [2]. Then, from eq. (24),  $g_{,kk} \equiv \nabla^2 \alpha = 0$ , quite generally.

Therefore, having determined  $\vec{\lambda}$  and having noted that  $g(x^k) = \alpha$ , we now turn to eqs. (18A) and (18B). For this purpose, rewrite eq. (18B) in the form

$$-\square^2 (\ell_o^2 \lambda_i) - 2 (p \ell_o^2)_{,i} - 2 p (\ell_o^2)_{,o} \lambda_i = \kappa \sigma \lambda_i$$

expand the D'Alembertian and use eqs. (18A) and (22) to obtain

$$m [g \nabla^2 \lambda_i + 2 \lambda_{i,k} g_{,k} - 2 (p g)_{,i}] + 2 p g [m_{,o} \lambda_i - m_{,i}] + 2 g \lambda_{i,k} m_{,k} = 0 \quad (29)$$

Then, we can use eq. (27) to simplify terms involving  $\lambda_{i,k}$  and reduce eq. (29) to

$$2 \alpha p \cos \vartheta [m_{,o} \lambda_i - m_{,i}] = 0 \quad (30)$$

where we have used  $g(x^k) = \alpha$  and  $\nabla^2 (\alpha \lambda_i) = (\alpha^2 + \beta^2)_{,i}$  as can be easily verified [2]. Now, multiplying and contracting eq. (30) with  $\lambda_i$ , we see that eq. (30) is equivalent to eq. (23).

We now use  $m_{,oo} - m_{,kk} = -m_{,o} \lambda_{k,k}$ , which follows from eq. (23), and substitute for  $\lambda_{k,k}$  from eq. (21B) to rewrite eq. (18A) as

$$\kappa \sigma = m_{,o} (\alpha_{,k} \lambda_k - 2 p \alpha) \quad (31)$$

But, differentiating eq. (27) with respect to  $x^k$  and manipulating the terms gives

$$\alpha_{,k} \lambda_k = \beta^2 - \alpha^2 = 2 p \alpha \cos \vartheta$$

and, hence, eq. (31) reduces to

$$\sigma = \frac{\alpha^2}{4\pi} m_{,o} \quad \zeta \equiv \frac{\sigma}{\ell_o^2} = \frac{\alpha}{4\pi} \frac{m_{,o}}{m} \quad (32)$$

Therefore, we have reduced the field equations, eqs. (18), to eqs. (23), (24) and (32).

It is important to note that by demanding in eq. (21A) that  $(L + A)/\ell_o$  to be independent of  $m$ , we have effectively decoupled eq. (20) from eqs. (30) and (32). That is to say, eqs. (30) and (32) are not required to determine the solution of eq. (20) which basically decides the symmetry of the solution. Further, we see that the equations yield the standard (Kerr) solution for  $m = \text{constant}$  [2].

We now turn to a specific solution to the field equations, eqs. (23), (24) and (32) that has the symmetry of the Kerr metric.

## 4 The Radiating and Rotating Metric

Let us consider eq. (24). As is well-known [2,3], the following is the solution of eq. (24) corresponding to the Kerr metric

$$\gamma = [x^2 + y^2 + (z - ia)^2]^{-1/2}$$

$$g(x^k) \equiv \alpha = \frac{\rho^3}{\rho^4 + a^2 z^2}, \quad \rho^2 = \frac{r^2 - a^2}{2} + \left[ \frac{(r^2 - a^2)^2}{4} + a^2 z^2 \right]^{1/2} \quad (33)$$

$$\lambda_x = \frac{\rho x + a y}{\rho^2 + a^2}, \quad \lambda_y = \frac{\rho y - a x}{\rho^2 + a^2}, \quad \lambda_z = \frac{z}{\rho}$$

where  $r^2 = x^2 + y^2 + z^2$  with  $x, y, z$  being the Cartesian coordinates and  $a$ , constant here, is the measure of the angular momentum per unit mass,  $m$ , of the central gravitating object. We note that in choosing the form of  $\gamma$  above, we have lost no generality [2]. As is well-known, this solution reduces to the spherically symmetric case when  $a = 0$ .



Further, it is easy to see that eq. (23) is equivalent to the existence of a new time coordinate, say,  $v$ , which satisfies

$$v_{,o} = v_{,i} \lambda_i \quad (34)$$

As is well-known, this is the relation satisfied by the choice  $v = t + \rho$ .

Now, in terms of the advanced null coordinate,  $v$ , eq. (32) is

$$\sigma = \frac{\alpha^2}{4\pi} m_{,v} \quad \zeta = \frac{\alpha}{4\pi} \frac{m_{,v}}{m} \quad m_{,v} \geq 0 \quad (35)$$

where  $m \equiv m(v)$  is an arbitrary, non-negative, increasing function of  $v$ . We note that eq. (35) then relates the density of radiation to the rate of change of mass,  $m$ , with respect to advanced null time  $v$  and can be taken as a defining relation. Therefore, we have completed our solution of the field equations, eqs. (18), or, equivalently, that of eqs. (23), (24) and (32).

Then, putting all the pieces together, the metric can be explicitly displayed in the  $(v, \rho, \theta, \phi)$  coordinates in the Carmeli-Kaye form as

$$\begin{aligned} ds^2 = & \left(1 - \frac{2m\rho}{\rho^2 + a^2 \cos^2 \theta}\right) dv^2 - (\rho^2 + a^2 \cos^2 \theta) d\theta^2 - 2dv d\rho \\ & - 2a \sin^2 \theta d\rho d\phi - \frac{4m\rho a \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta} dv d\phi \\ & - \left[\rho^2 + a^2 + \frac{2m\rho a^2 \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta}\right] \sin^2 \theta d\phi^2 \end{aligned} \quad (36)$$

where  $m \equiv m(v)$  and we have changed the coordinates as

$$\begin{aligned} \cos \theta &= \frac{z}{\rho} \quad x + iy = (\rho - ia) e^{i\phi} \sin \theta \\ d\tilde{\phi} &= d\phi + \frac{2a}{\rho^2 + a^2} d\rho \end{aligned} \quad (37)$$

and dropped the tilde over  $\phi$  while displaying the metric. Clearly, for  $m = \text{constant}$ , this is the Kerr metric. And, for  $m \equiv m(v)$  but  $a = 0$ , this reduces to the Vaidya metric. In this last case,  $\alpha = \frac{1}{r}$  and, hence,  $\sigma = m_{,v}/4\pi r^2$  while  $\zeta = m_{,v}/4\pi r m$ . Clearly, it is  $\sigma$  that is directly related to the flux of radiation.

We note that since  $m_{,v} \geq 0$  the metric of eq. (36) corresponds to a situation of collapsing radiation shells. Then, the metric corresponding to expanding radiation shells can be obtained if we choose  $\ell_\alpha = \ell_o (1, -\vec{\lambda})$  in the place of  $\ell_\alpha$  as in eq. (17). In this case, we shall then obtain  $m_{,u} \leq 0$  where  $u = t - \rho$  is the retarded null time. Of course, the metric of eq. (36) will then have to be written in terms of the new  $(u, \rho, \theta, \phi)$  coordinates accordingly but we shall not write that form of the metric explicitly here.

## 5 Concluding Remarks

In conclusion, we have obtained here a non-static solution of the field equations of General Relativity representing the gravitational field of axisymmetric radiation flow beginning with a metric of the Eddington or the Kerr-Schild form. The solution manifestly possesses the symmetry of the Kerr metric.

## References

- [1] P C Vaidya (1951) *Proc. of the Indian Acad. Sci.* **A33**, 264.
- [2] R Adler, Bazin, M. and Schiffer, M. (1975) *Introduction to General Relativity* (McGraw Hill-Kogakusha, Tokyo).  
  
S Chandrasekhar (1983) *Mathematical Theory of Black Holes* (Clarendon Press, Oxford).
- [3] R P Kerr (1963) *Phys. Rev. Lett.* **11**, 237.
- [4] P C Vaidya and L K Patel (1973) *Phys. Rev.* **D7**, 3590 and references therein.  
  
M Carmeli and M Kaye (1977) *Ann. Phys. (N.Y.)* **103**, 97 and references therein.  
  
D Kramer and U Hähner (1995) *Class. Quantum Grav.* **12**, 2287 and references therein.
- [5] R Penrose (1978) in *Theoretical Principles in Astrophysics and Relativity* edited by N R Liebowitz, W H Reid and P O Vandervoort (University of Chicago Press, Chicago).
- [6] S M Wagh and P S Muktibodh (1998) submitted. CIRI-98-GR-2.  
  
S M Wagh and P S Muktibodh (1998) submitted. CIRI-98-GR-3.  
  
S M Wagh, P S Muktibodh and R V Saraykar (1998) in preparation.